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Symbol	LATEX	Code Point
∪	\cup	8746
∩	\cap	8745
≠	\neq	8800
≥	\geq	8805
≤	\leq	8804
=	\Equal	_61
≈	\approx	8776

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UNIVERSITY OF ILLINOIS
GRADUATE COLLEGE
DIGITAL COMPUTER LABORATORY

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THEORY OF ASYNCHRONOUS CIRCUITS

BY David C. Muller

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THEORY OF ASYNCHRONOUS CIRCUITS

I. Introduction

Asynchronous circuits have been loosely defined as switching circuits which do not require a "clock" or source of fundamental frequency for their operation. A considerable portion of the logical circuitry in *Illiac* requires no clock signals and may therefore be considered asynchronous. One advantage of asynchronous circuitry is the additional speed one obtains by allowing each operation to follow the one preceding it without having to wait for the clock signal to occur. Another advantage results from the fact that if asynchronous circuits are constructed according to certain rules they will not proceed incorrectly if one of the elements fails to act but will merely stop. The faulty element may then be located by observing the state of the circuit. This latter advantage will be explained in more detail in the later sections of this discussion.

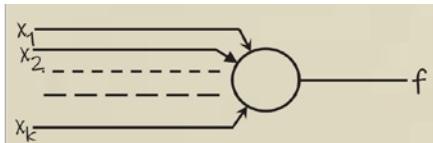
One might think that in order to design asynchronous circuits one would have to keep close account of the times taken by the various elements in acting so that the entire circuit would behave in the way intended. This is true in some designs but it is possible to design circuits in such a way that the relative speeds of the elements do not affect the over-all behavior of the circuit. Such designs are of particular interest to us and are the ones which we shall analyze.

In dealing with asynchronous circuits one finds that most of the definitions and concepts which have been developed for synchronous, or clocked, circuits no longer apply and it is necessary to begin at the beginning and redefine everything.

II. Preliminary Considerations

Definition 1. Decision Element

A decision element has one output line f and a specified number k of input lines x_1, x_2, \dots, x_k . It may be represented by a circular symbol in the following way.



At any given time a signal having a value either 0 or 1 must appear on each of the $k+1$ lines. The element is said to be in equilibrium for certain combinations of values appearing at the points f and x_1, x_2, \dots, x_k and is said to be excited for all other combinations. The list of the former combinations is given by the specifications of the element. Any change occurring in f must be such as to place the element in equilibrium from an excited condition. If the inputs are held unchanged from some time t_0 on and the element is excited then a change in f will occur at some time $t_0 + T$ where T is bounded by $0 < T < M$. M is an upper bound on all T but T itself is otherwise indefinite and may vary in any way from one element to another and even for a given element from one change to the next.

We notice that the above definition may be self-contradictory if for a set of values at the inputs both 0 and 1 at line f represent excited states. In this case the first part of the definition prohibits any change from occurring while the second part tells us that it must occur. We shall avoid this difficulty by never applying input combinations to elements which admit no equilibrium condition.

It is interesting to contrast the above definition of a decision element with that which is used for synchronous circuits. In synchronous circuits one usually assumes a fixed time of operation for each el-

ement. This assumption permits the use of delay operations and other aids in the analysis of circuits which cannot be applied to asynchronous circuits.

Definition 2. Asynchronous Circuit

An asynchronous circuit is an interconnection of decision elements. This interconnection is done by attaching the lines together at points called nodes. Each node has at least one line connected to it and no more than one decision element output connected to it. No delay is assumed to take place along lines in the circuit.

Definition 3. Complete Circuit

A complete circuit is an asynchronous circuit in which each node has a decision element output connected to it. Each node is then said to belong to the decision element which feeds it.

III. The Notion of Speed Independence

We mentioned earlier that we wish to treat circuits whose over-all behavior is independent of action times of the elements which make them up. This loose definition must be formalized so that it is applicable to circuits of the types defined earlier.

Definition 4. Immediate State of a Circuit

The immediate state of an asynchronous circuit (abbreviated I-state) is given by specifying the value of the signal at each node i n the circuit.

Definition 5. Equilibrium State of a Circuit

An asynchronous circuit is said to be in equilibrium if every element in the circuit is in equilibrium.

Condition 1: A complete circuit satisfies condition 1 with respect to an I-state S if when placed in state S it will, after sufficient time, arrive at a unique equilibrium state C regardless of the action times taken by the elements.

This definition seems to satisfy our intuitive notion which we referred to before and is essentially the same as the condition given by Huffman [1]. It has two main weaknesses, however.

- (1) It cannot be applied to circuits which cycle indefinitely, and we should like to apply our notions to such circuits.
- (2) It is a very difficult condition to apply to an actual circuit since one would have to treat every possible I-state which might occur during the action of the circuit.

Condition 2: A complete circuit satisfies condition 2. (We shall say it is totally sequential.) with respect to an I-state S if when placed in state S it will go through a unique sequence of I-states S_1, S_2, \dots , either indefinitely or until it arrives at an equilibrium state C.

This condition also has been treated by Huffman and others [2], and it certainly avoids the two objections listed above. We shall see, however, that it implies that only one element can be excited at a time in the circuit. This restriction is far too severe to be tolerated since it removes from consideration all circuits in which parallel action occurs, such as in gates, registers, and adders. If speed independence requires the penalty of serial action and the resulting low duty cycle then let us turn to other types of circuits

Definition 6. Broken Complete Circuit

A complete circuit is said to be broken at a given node if that node is disconnected from the decision element which feeds it. If a circuit is broken when it is in a given state an artificial signal is placed at

the node where it is broken which holds that node at the value specified by the state. A circuit may be broken at more than one node.

Condition 3: A complete circuit satisfies condition 3 with respect to an I-state S if all possible I-states U which the circuit may go through after being placed in S have the following property:

If the circuit is broken at any set of nodes (the circuit may be broken at no node, all nodes, or any combination of the nodes) when in state U then depending on U and the set of nodes at which the circuit is broken it will either cycle indefinitely, or it will proceed to an equilibrium state C which does not depend on the relative speeds of the elements.

This condition should allow one to treat circuits which cycle indefinitely and yet it is not as restrictive as condition 2. It still suffers from the disadvantage that it is a difficult condition to apply.

Theorem 1: If a circuit starting in I-state S does not cycle indefinitely then condition 3 implies condition 1.

Proof: Since the circuit does not cycle indefinitely it must reach an equilibrium state E . Let us choose $U = S$ and break none of the nodes in the circuit. Then by condition 3 E is unique and condition 1 is satisfied.

Theorem 2: Condition 2 is equivalent to the statement that only one decision element can be excited at a time.

Proof: Assume that in some I-state U more than one decision element is excited. Then depending upon which decision element goes to equilibrium first we may have any one of several nodes changing after state U . This means that any one of several states may follow U . Assume now that only one decision element is excited in state U . Then only one node can change (the excited one) and U must be followed by a unique state.

Theorem 3: Condition 2 implies condition 3.

Proof: Let us first consider the effect of breaking a node whose decision element is to become excited. When the signal at the output of the decision element changes, the node will remain unaffected, since the break has disconnected the decision element from its node. We see therefore that if condition 2 applies and we break the circuit in a set of nodes when it is in state U it will continue to pass through a unique set of states (the same as it would if the nodes were not broken) until the decision element feeding a broken node becomes excited. If this happens the circuit will pass to equilibrium since by theorem 2 this element is the only excited one and when it goes to equilibrium no nodes will be affected and the entire circuit will be in equilibrium. Since the sequence of states is unique an equilibrium state will be unique if it is reached.

The notion of a broken circuit and the statement of condition 3 give us a better idea of what is needed for speed independence in a circuit. It turns out, however, that condition 3 is difficult to apply to actual circuits and also very difficult to treat theoretically. For this reason we shall strengthen condition 3 very slightly and set down a condition which also applies to most practical cases and which is considerably more convenient.

Condition 4: Speed Independence. A complete circuit is said to satisfy condition 4 (is speed independent) with respect to an I-state S if during all possible transitions which follow S no decision element passes from an excited state to an equilibrium state unless in doing so its output changes; that is to say, unless the decision element itself acts.

In the theory which follows we shall use condition 4 (speed independence) as our basic condition and derive its properties. We shall also show toward the end of this development that it implies condition 3.

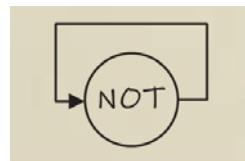
IV. Properties of Speed Independent Circuits

Definition 7. Cumulative State (C-state)

A cumulative state or C-state is defined with respect to some initial I-state S by listing the numbers of signal changes which have occurred at the nodes since the circuit was placed in I-state S. The C-state is thus an n-dimensional vector (if there are n nodes) whose components are all non-negative integers.

It should be noted that the C-state provides information concerning the history of the circuit while the I-state is a vector of 0's and 1's which merely describes the momentary condition of the circuit. In an I-state U the I-state equals the sum of the C-state, and S modulo 2 where the sum is taken component-wise. Thus s and the C-state uniquely determine the I-state U whereas several C-states may correspond to the same I-state. This would mean that U may be obtained in several ways.

To illustrate the difference between the two types of states consider the following one node circuit:



The I-states follow the sequence 0, 1, 0, 1 while the C-states follow the sequence 0, 1, 2, 3, In this case the state vectors have but one component.

Definition 8. Directly Following

An I- or C-state B is said to directly follow a state A if B results from A by a change in a single node and if the decision element for this node is excited in state A.

We note that if more than one element is excited in state A then B will follow A only if the excited element whose change leads to B acts first. Thus we see that "B directly follows A" means that B may actually follow A in time but will only follow it with certainty if but one element is excited in state A.

Definition 9. State Ordering, e.g., $A \leq B$

Given two C-states A and B shall write $A \leq B$ if each component of A is no greater than the corresponding component of B.

Theorem 4: If B may follow A in time, in any asynchronous circuit, then $A \leq B$.

Proof: If B may follow A in time then either $A = B$ or some nodes have undergone changes in passing from A to B. These components will have increased and in either case $A \leq B$.

Theorem 5: The set of C-states which may follow an initial state S in any asynchronous circuit are partially ordered in time.

Proof: We must show that the three partial ordering relations are satisfied with respect to the relation “may follow.”

- (1) A may follow A.

This is true since no element need act and A passes to itself.

- (2) If B may follow A and A may follow B then $A = B$.

This results from theorem 4 since we have $A \leq B$ and $B \leq A$. Hence $A = B$ componentwise.

- (3) If B may follow A and C may follow B then C may follow A.

This is true since one way of passing from A to C is through the intermediate C-state B.

Theorem 5 applies only to C-states and indeed the property of partial ordering seems to reflect the special definition of the C-state rather than any inherent property of the circuit. It will be needed, however, in all the later results.

Definition 10. Union of States, $A \cup B$

If A and B are two C-states then $A \cup B$ is a vector whose components are the maxima of the corresponding components of A and B. This vector may or may not correspond to an actual C-state which may occur during the operation of the circuit.

Lemma: In a speed independent circuit, if B results from A with no intermediate C-states, then there is a sequence of C-states A, A_1, A_2, \dots, B such that each member of the sequence directly follows the one preceding it in the sequence.

Proof: In going from A to B a set of one or more decision elements must act simultaneously. If only one element acts then B directly follows A and the result is trivial. If more than one decision element acts then all those which act must be excited in state A. We should note here that the time T for any decision element to act is greater than 0 by the definition so that if several decision elements act simultaneously they must all be excited. By condition 4 they might instead act separately in any order since each must act to reach equilibrium. If this were to occur we would get the desired sequence A, A_1, A_2, \dots, B .

Theorem 6: In a speed independent circuit if B may follow A then there is a sequence of C-states A, A_1, A_2, \dots, B such that each member of the sequence directly follows the one preceding it in the sequence.

Proof: Since B may follow A there is some sequence of states which the circuit may pass through in going from A to B. Between any pair of these we may substitute a sequence of states which directly follow each other. If this is done for each pair we obtain the desired sequence.

Corollary: B directly follows A if and only if there exists no state C such that $B \geq C \geq A$.

Proof: The previous theorem combined with theorem 4 yields this result. The second condition will be written B covers A.

Theorem 7: In a speed independent circuit, if two different C-states X and Y both directly follow A then $X \cup Y$ exists as a possible state and it directly follows both X and Y.

Proof: Let us assume that X results from A by a change in the i'th node and that Y results from A by a change in the j'th node. We see that $i \neq j$ since X and Y are not the same state. This means that in state A both i and j are fed by excited decision elements. In state X the j'th element is still excited by condition 4 and hence the j'th node may change and give state $X \cup Y$. Thus $X \cup Y$ directly follows X. Similarly it directly follows Y and the theorem is proved.

Lemma: In a speed independent circuit if C-state X directly follows A and if B may follow A then $X \cup B$ exists and may follow X and B.

Proof: Since B may follow A we may construct a sequence of states by theorem 6 A, A_1, A_2, \dots, B such that each member directly follows the one preceding it. Two cases may now occur.

- (1) $A_1 = X$. In this case $X \cup A_1 = A_1 = X$ by definition 10, so B may follow X giving $X \cup B$ and $X \cup B = B$ thus proving the lemma.
- (2) $A_1 \neq X$. In this case A_1 and X both directly follow A and theorem 7 applies. Thus $X \cup A_1$ exists and directly follows both A_1 and X. In the above argument we may now replace A by A_1 and X by $X \cup A_1$. The only difference is that the sequence A_1, A_2, \dots, B is shorter than A, A_1, A_2, \dots, B by one state.

Let us assume that we repeat the application of (2) until the i 'th step. We shall now have formed $((X \cup A_1) \cup A_2) \dots \cup A_i = X \cup A_i$. This state may follow both X and A_i . If $X \cup A_i = A_{i+1}$ then B may follow X and the lemma proved as in (1). If (1) never occurs then $((X \cup A_1) \cup A_2) \dots \cup B = X \cup B$ will be formed and it will follow both X and B (by theorem 7) thus also proving the lemma.

Theorem 8: In a speed independent circuit for any C-states X and Y the C-state $X \cup Y$ exists and may follow both X and Y.

Proof: Since the initial state S may be followed by X and also may be followed by Y we may construct two sequences as in theorem 6

$$\begin{aligned} &S, X_1, X_2, \dots, X \\ &S, Y_1, Y_2, \dots, Y \end{aligned}$$

By the lemma we form $X \cup Y_1$. This may follow Y_1 so we can also construct $(X \cup Y_1) \cup Y_2$. Similarly we form $((X \cup Y_1) \cup Y_2) \dots \cup Y = X \cup Y$. This may follow both X and Y and the theorem is proved.

Corollary: In a speed independent circuit if A and B are C-states and $A \geq B$ then A may follow B. This is the converse of theorem 4.

Proof: Since $A \geq B$ we have $A \cup B = A$ and $A \cup B$ may follow B.

This last result, together with theorem 4, means that $A \geq B$ is equivalent to A may follow B in a speed independent circuit.

Theorem 9: In a speed independent circuit if $X \leq A$ and $Y \leq A$ then $X \cup Y \leq A$.

Proof: This result follows from the definitions in terms of numerical vectors and shows that $X \cup Y$ is a least upper bound of X and Y.

Theorem 10: In a speed independent circuit for any two C-states X and Y there exists a unique state A such that

$$\begin{aligned} A \leq X &\quad \text{and} \quad A \leq Y \quad \text{and if for any } B \\ B \leq X &\quad \text{and} \quad B \leq Y \quad \text{then } B \leq A. \end{aligned}$$

We shall use the notation $X \cap Y$ for A.

Proof: There is at least one state R such that $R \leq X$ and $R \leq Y$ since the initial state S is such a state. The number of such states is finite, however, since X and Y are finite vectors. Let us write these $R_1, R_2,$

..., R_k . Now we form $R_1 \cup R_2 \cup \dots \cup R_k = A$. Clearly $R_i \leq A$ for any R_i but from the numerical definitions we also have $A \leq X$ and $A \leq Y$.

Theorem 11: The C-states of a speed independent circuit which may follow the initial state S form a *semi-modular lattice* [3] with respect to the symbols \cap , \cup , \leq .

Proof: Theorems 5, 9, and 10 prove that the states form a lattice. Theorem 7 and the corollary to 6 prove that it is semi-modular. We may note here also that this lattice must contain a zero element (the state S) since the state S may be followed by any other state.

In most switching circuit applications one is only concerned with the signals appearing at some of the nodes of the circuit. The other nodes may be needed to allow the construction of the circuit from a given set of elements but the signals on them will not otherwise be of interest. Let the interesting set of nodes by N_1, N_2, \dots, N_m . If in a given C-state the numbers corresponding to these nodes are a_1, a_2, \dots, a_m we may regard this set of numbers in much the same way as we regarded the complete set of numbers corresponding to the C-state. This leads us to

Definition 11. A Break Set of States

A break set of states consists of all the C-states for which the numbers on nodes N_1, N_2, \dots, N_m (a subset of all n nodes) are no greater than a_1, a_2, \dots, a_m .

A break set is the set of states which would be obtained if each node N_i were broken when its number became a_i . This would prevent its number from

becoming greater than a_i and yet each state in the break set may occur since it follows from the initial state S without signals having to pass broken nodes.

Theorem 12: A break set in a speed independent circuit is an ideal of the lattice Of C-states.

Proof: If X and Y are in the break set then $X \cup Y$ is in the break set. For any T we have $T \cap X \leq X$ and hence also in the break set.

This theorem is important since it permits us to treat a subset of the nodes in the circuit in many ways as if it were the complete set. The break sets now correspond to the states and the algebra of ideals corresponds to the lattice algebra.

Theorem 13: Speed independence implies condition 3.

Proof: Let us break our circuit, when it is in state U, at nodes N_1, N_2, \dots, N_m . Let us also assume that the numbers on these nodes are a_1, a_2, \dots, a_m . Let the circuit go to the equilibrium state J. We wish to show that J is unique. Assume that the circuit might also go to state K such that J cannot follow K. Then $J \cup K$ is also in the break set and $J \cup K \leq J$ so that J cannot be an equilibrium state.

We see furthermore that if J does represent an equilibrium state then the ideal associated with the break set must be a principal ideal.

v. Composition of Speed Independent Circuits

It is of particular importance to the circuit designer to be able to form speed independent circuits in some systematic way. This composition of circuits is done in the synchronous case by putting together several partial circuits which are designed individually. In the asynchronous case the problem is more difficult since an arbitrary combination of speed independent circuits will not be speed independent, in

general. For this reason we must develop rules for combination of speed independent circuits which preserve this property. The rules so far developed are all based on the following theorems.

Theorem 14: If a speed independent circuit is broken at a node N_i the output of the decision element feeding this node can change at most once after the break has been made.

Proof: Let us assume that the output of the decision element feeding the broken node can change twice (or more times) after the break has occurred. Since the break prevents the change in output from affecting the nodes in the circuit in any way we see that if the decision element had been slower and the first change had not occurred then the changes in the inputs to this element which produced the second change would have occurred anyhow and the element would have gone from an excited state to an equilibrium state without having acted. This would constitute a violation of condition 4.

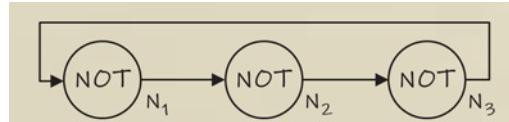
Theorem 15: Let a “not” element in a speed independent circuit be replaced by a second speed independent circuit which is broken at one node N_1 . Let the broken node be fed by the input for the “not” element and the output of the element which fed the broken node replace the output of the “not” element. If this replacement is made when the states in the two circuits are such that the signals match at the points which are joined then the resulting combined circuit is speed independent.

Proof: Two cases must be considered:

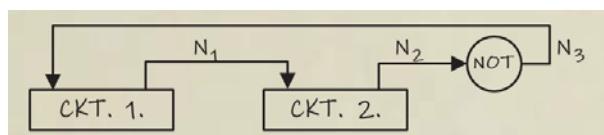
- (1) Let us assume that the “hot” element is in equilibrium when it is replaced. This means that in order to match signals the broken circuit must have had the element act once since the break was made. This means that it will not act a second time provided the signal at N_1 is not changed.
- (2) Let us assume that the “not” element is excited when it is replaced. This means that the element has not acted yet. If it does act this will affect the first circuit in the same way that an action of the ‘hot’ element would have affected it. If it does not act then the first circuit will behave as if the output of the “not” circuit had been broken. Thus we see that the first circuit will not lose the property of speed independence. The second circuit also retains speed independence for the signal feeding N_1 cannot change more than once after the element acts since if it were to change twice or more this would mean that the signal on the “not” circuit could have changed twice with no change in its output this bringing the “not” circuit back into equilibrium without its acting and violating condition 4.

We now give several examples to show how theorem 15 may be used to build up complex circuits from simpler ones.

Example 1: The circuit

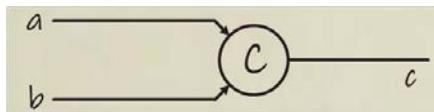


is speed independent, provided the signals on all three nodes are not the same. Therefore we can replace two of the “nots” with arbitrary speed independent circuits, each broken in one node.



In this combination the two circuits alternate in having the signals at the broken nodes change as long as these changes continue to occur. In this arrangement we may, therefore, regard the two circuits as acting alternately.

Now let us define the C-element in the following way:

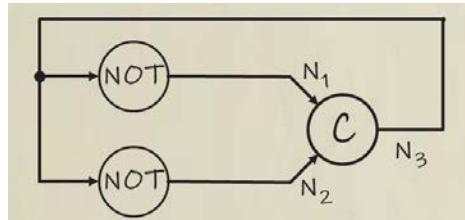


Equilibrium states are:

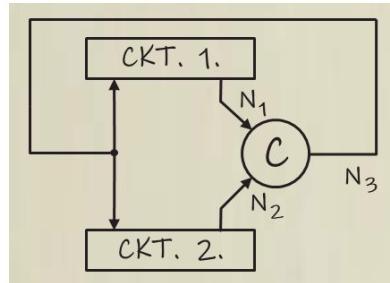
a	b	c
0	0	0
1	1	1
0	1	0
0	1	1
1	0	0
1	0	0

In other words the output will tend toward the value of the two inputs whenever they agree. Otherwise the output will remain at its previous value. Using this element we form

Example 2: The circuit



is speed independent regardless of its initial state. Changes occurring at N_1 and N_2 may bear any time relation to each other but the change at N_3 will not occur until both N_1 and N_2 have changed and agree. We may now replace the two "not" elements by arbitrary broken circuits thus



These two circuits may now be thought of as acting in parallel. The change in the faster of the two circuits will be held up until the slower has acted.

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